MATH 2700 CURVE ANALYSIS: DEFINITIONS, DERIVATIONS, AND PROOFS (OH MY!)

- $\vec{r}(t)$ is the **position** function; $\vec{r}(t)$ tells where you are at time t.
- $\vec{v}(t) = \vec{r}\,'(t)$ is the **velocity** function; $\vec{v}(t)$ tells you how fast and in which direction you're moving.
- $\|\vec{v}(t)\|$ is the **speed**; $\|\vec{v}(t)\|$ tells you how fast you're going.

  NOTE: Think of $\|v(\vec{t})\|$ as the 'speedometer' function.
- $\hat{T}(t) = \dfrac{\vec{v}(t)}{\|\vec{v}(t)\|}$ is the **principal unit tangent vector**; $\hat{T}(t)$ tells you the direction you're going.

  NOTE: For $\hat{T}(t)$ to exist, $\|v(\vec{t})\| \neq 0$. In this case, $\vec{v}(t) = \|\vec{v}(t)\|\,\hat{T}(t)$.
- $\vec{r}(t)$ is **smooth** if $\vec{r}\,'(t) = \vec{v}(t) \neq \vec{0}$ or, equivalently, $\|\vec{r}\,'(t)\| = \|\vec{v}(t)\| \neq 0$.
- $s(t) = \displaystyle\int_a^t \|\vec{v}(u)\|\,du$ is the **arc length parameter**. It tells you how far you've travelled over $[a, t]$.

  NOTE: Think of $s(t)$ as the 'odometer' function and note that: $\dfrac{ds}{dt} = \|\vec{v}(t)\|$.
- $\hat{N}(t) = \dfrac{\hat{T}\,'(t)}{\|\hat{T}\,'(t)\|}$ is the **principal unit normal vector**; $\hat{N}(t)$ tells you the direction you're turning.

  NOTE: For $\hat{N}(t)$ to exist, $\|\hat{T}\,'(t)\| \neq 0$; i.e., $\hat{T}(t)$ needs to be smooth for $\hat{N}(t)$ to exist.
- $\hat{B}(t) = \hat{T}(t) \times \hat{N}(t)$ is the **principal unit binormal vector**.

  NOTE: $\hat{B}(t)$ is orthogonal to both $\hat{T}(t)$ and $\hat{N}(t)$ and $\|\hat{B}(t)\| = 1$.
- The **Frenet Frame** or **TNB-frame** is 3-D coordinate system determined by $\hat{T}(t)$, $\hat{N}(t)$, and $\hat{B}(t)$.
  - The **osculating plane** is the plane determined by $\hat{T}(t)$ and $\hat{N}(t)$ with normal vector $\hat{B}(t)$.
  - The **normal plane** is the plane determined by $\hat{N}(t)$ and $\hat{B}(t)$ with normal vector $\hat{T}(t)$.
  - The **rectifying plane** is the plane determined by $\hat{T}(t)$ and $\hat{B}(t)$ with normal vector $\hat{N}(t)$.
- $\vec{a}(t) = \vec{r}\,''(t) = \vec{v}\,'(t)$ is the **acceleration function**.

  NOTE: $\vec{F}(t) = m\vec{a}(t)$, so $\vec{a}(t)$ is a scalar multiple of the force which keeps you on the path $\vec{r}(t)$.
- $\kappa(s) = \left\|\dfrac{d\hat{T}}{ds}\right\| = \left\|\hat{T}\,'(s)\right\| = \dfrac{d\hat{T}}{ds} \cdot \hat{N}$ is the **curvature** function.

  NOTE: $\kappa(s)$ measures how fast you're turning as a function of how far you've traveled, or, alternatively, the tendency to 'twist' out of the normal plane.
- $\tau(s) = -\dfrac{d\hat{B}}{ds} \cdot \hat{N}(s)$ is the **torsion** function.

  NOTE: $\tau(s)$ measures the rate at which the motion is 'twisting' out of the osculating plane.

THEOREMS:

1. **CIRCULAR / SPHERICAL MOTION:** If $\|\vec{r}(t)\| = R$, then $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$; that is, $\vec{r}(t) \perp \vec{r}\,'(t)$.

   PROOF: $\vec{r}(t) \cdot \vec{r}(t) = \|\vec{r}(t)\|^2 = R^2$, so differentiating using the product rule we get:

$$\begin{aligned} D_t\left[\vec{r}(t) \cdot \vec{r}(t)\right] &= D_t[R^2] \\ \vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t) &= 0 \\ 2\,\vec{r}(t) \cdot \vec{r}\,'(t) &= 0 \\ \vec{r}(t) \cdot \vec{r}\,'(t) &= 0 \end{aligned}$$

   Hence, $\vec{r}(t) \perp \vec{r}\,'(t)$.

   NOTE: if $\|\vec{r}(t)\| = R$, then the motion occurs on a circle (sphere) of radius R.

2. **THEROEM:** $\hat{T}(t) \perp \hat{N}(t)$: If $\hat{N}(t)$ exists, then $\hat{T}(t) \cdot \hat{N}(t) = 0$.

   PROOF: Since, by definition, $\|\hat{T}(t)\| = 1$, we have by the previous result that $\hat{T}(t) \cdot \hat{T}\,'(t) = 0$. Since $\hat{N}(t)$ is a scalar multiple of $\hat{T}\,'(t)$, we are done.

   NOTE: Hence the use of the word 'normal' in describing $\hat{N}(t)$ as the principal unit **normal** vector.

3. **COMPONENTS OF ACCELERATION:**

   If $\vec{r}(t)$ and $\hat{T}(t)$ are smooth, then there are functions $a_T(t)$ and $a_N(t)$ so that:

$$\vec{a}(t) = a_T(t)\hat{T}(t) + a_N(t)\hat{N}(t).$$

   PROOF:

$$\begin{aligned} \vec{a}(t) &= D_t[\vec{v}(t)] \\ &= D_t[\|\vec{v}(t)\|\,\hat{T}(t)] \\ &= D_t[\|\vec{v}(t)\|]\,\hat{T}(t) + \|\vec{v}(t)\| D_t[\hat{T}(t)] \\ &= D_t\left[\|\vec{v}(t)\|\right]\,\hat{T}(t) + \|\vec{v}(t)\|\,\hat{T}\,'(t) \\ &= D_t\left[\|\vec{v}(t)\|\right]\,\hat{T}(t) + \|\vec{v}(t)\|\,\|\hat{T}\,'(t)\|\hat{N}(t). \end{aligned}$$

   We define $a_T(t) = D_t\left[\|\vec{v}(t)\|\right]$ and $a_N(t) = \|\vec{v}(t)\|\,\|\hat{T}\,'(t)\|$, which proves the claim.

   NOTE:

   - The vector $a_T(t)\hat{T}(t)$ is the **tangential component of the acceleration**; $a_T(t)$ describes how much acceleration is in the direction you're traveling.
   - The vector $a_N(t)\hat{N}(t)$ is the **normal vector component of acceleration**; $a_N(t)$ describes how much of the acceleration is in the direction you're turning. Note that $a_N(t) \geq 0$.
   - We now know the acceleration (hence force) vector lives entirely in the osculating plane.

For the remainder of this handout, we will suppress the functional dependence on t and implicitly assume whatever smoothness conditions are needed for the vector valued functions to be defined.

4. **THEOREM:** $\|\vec{a}\|^2 = (a_T)^2 + (a_N)^2$.

PROOF:

$$\begin{aligned}
\|\vec{a}\|^2 &= \vec{a} \cdot \vec{a} \\
&= \left(a_T \hat{T} + a_N \hat{N}\right) \cdot \left(a_T \hat{T} + a_N \hat{N}\right) \\
&= \left(a_T \hat{T}\right) \cdot \left(a_T \hat{T}\right) + \left(a_T \hat{T}\right) \cdot \left(a_N \hat{N}\right) + \left(a_N \hat{N}\right) \cdot \left(a_T \hat{T}\right) + \left(a_N \hat{N}\right) \cdot \left(a_N \hat{N}\right) \\
&= (a_T)^2 \left(\hat{T} \cdot \hat{T}\right) + a_T a_N \left(\hat{T} \cdot \hat{N}\right) + a_N a_T \left(\hat{N} \cdot \hat{T}\right) + (a_N)^2 \left(\hat{N} \cdot \hat{N}\right) \\
&= (a_T)^2 + (a_N)^2 .
\end{aligned}$$

NOTE: This result extends to any (family) of (pairwise) orthogonal unit vectors.

5. **ALTERNATE FORMULAS FOR a_T:**

$$a_T = \vec{a} \cdot \hat{T} = \frac{\vec{v} \cdot \vec{a}}{\|\vec{v}\|} = \frac{d^2 s}{dt^2}$$

PROOF: Take the equation $\vec{a} = a_T \hat{T} + a_N \hat{N}$ and 'dot' both sides with $\hat{T}$ to get:

$$\begin{aligned}
\vec{a} \cdot \hat{T} &= \left(a_T \hat{T} + a_N \hat{N}\right) \cdot \hat{T} \\
&= \left(a_T \hat{T}\right) \cdot \hat{T} + \left(a_N \hat{N}\right) \cdot \hat{T} \\
&= a_T \left(\hat{T} \cdot \hat{T}\right) + a_N \left(\hat{N} \cdot \hat{T}\right) \\
&= a_T ,
\end{aligned}$$

proving the first equality. For the second equality, substitute $\hat{T} = \dfrac{\vec{v}}{\|\vec{v}\|}$ into the first.

For the third equality, by definition, $a_T = D_t[\|\vec{v}\|]$ and $\|\vec{v}\| = \dfrac{ds}{dt}$, so

$$a_T = D_t \left[\frac{ds}{dt}\right] = \frac{d^2 s}{dt^2}.$$

6. **ALTERNATE FORMULAS FOR CURVATURE:**

$$\kappa = \frac{\|T'(t)\|}{\|\vec{v}(t)\|} = \frac{d\hat{T}}{ds} \cdot \hat{N}(s)$$

Since $\dfrac{ds}{dt} = \|\vec{v}(t)\| > 0$, we have that $s(t)$ is one-to-one. Hence, $\dfrac{dt}{ds} = \dfrac{1}{(ds/dt)} = \dfrac{1}{\|\vec{v}(t)\|}$.

Using the chain rule,

$$\kappa = \left\|\frac{d\hat{T}}{ds}\right\| = \left\|\frac{d\hat{T}}{dt}\frac{dt}{ds}\right\| = \left\|T'(t)\frac{1}{\|\vec{v}(t)\|}\right\| = \frac{\|T'(t)\|}{\|\vec{v}(t)\|}.$$

By definition, $\hat{N}(s) = \dfrac{\hat{T}'(s)}{\|\hat{T}'(s)\|} = \dfrac{\hat{T}'(s)}{\kappa} = \dfrac{1}{\kappa}\dfrac{d\hat{T}}{ds}$, or $\kappa\hat{N}(s) = \dfrac{d\hat{T}}{ds}$. If we 'dot' both sides with $\hat{N}$:

$$\kappa = \kappa(1) = \kappa\left(\hat{N}(s) \cdot \hat{N}(s)\right) = \frac{d\hat{T}}{ds} \cdot \hat{N}(s)$$

7. **ALTERNATE FORMULAS FOR a_N:**

$$a_N = \vec{a} \cdot \hat{N} = \|\vec{a} \times \hat{T}\| = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|} = \kappa\left(\frac{ds}{dt}\right)^2 = \sqrt{\|\vec{a}\|^2 - a_T^2}.$$

PROOF: For the first equality, repeat the argument in #5, 'dotting' with $\hat{N}$ instead of $\hat{T}$.

To get the second inequality, we 'cross' the equation $\vec{a} = a_T\hat{T} + a_N\hat{N}$ with $\hat{T}$:

$$\begin{aligned}
\vec{a} \times \hat{T} &= \left(a_T\hat{T} + a_N\hat{N}\right) \times \hat{T} \\
&= \left(a_T\hat{T}\right) \times \hat{T} + \left(a_N\hat{N}\right) \times \hat{T} \\
&= a_T\left(\hat{T} \times \hat{T}\right) + a_N\left(\hat{N} \times \hat{T}\right) \\
&= a_N\left(\hat{N} \times \hat{T}\right)
\end{aligned}$$

Hence taking the magnitudes of both sides, and remembering that $a_N \geq 0$, we get:

$$\|\vec{a} \times \hat{T}\| = \left\|a_N\left(\hat{N} \times \hat{T}\right)\right\| = \|a_N\|\,\|\hat{N} \times \hat{T}\| = a_N.$$

To get the third equality, substitute $\hat{T} = \dfrac{\vec{v}}{\|\vec{v}\|}$ into the second equality and use properties of $\times$.

(In particular, note since $\vec{a} \times \vec{v} = -\vec{v} \times \vec{a}$, $\|\vec{a} \times \vec{v}\| = \|\vec{v} \times \vec{a}\|$.)

To get the fourth equality, we use the definition of $a_N = \|\vec{v}(t)\|\,\|\hat{T}'(t)\|$:

$$a_N = \|\vec{v}(t)\|\,\|\hat{T}'(t)\| = \|\vec{v}(t)\|^2\,\frac{\|\hat{T}'(t)\|}{\|\vec{v}(t)\|} = \left(\frac{ds}{dt}\right)^2\kappa = \kappa\left(\frac{ds}{dt}\right)^2.$$

Finally, since $\|\vec{a}\|^2 = a_T^2 + a_N^2$, $a_N = \sqrt{\|\vec{a}\|^2 - a_T^2}$, since $a_N \geq 0$.

8. **ALTERNATE FORMULA FOR CURVATURE (REPRISE):**

$$\kappa = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|^3}$$

From our work above, we have: $a_N = \dfrac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|} = \kappa \left(\dfrac{ds}{dt}\right)^2 = \kappa\|\vec{v}\|^2$, so we divide through by $\|\vec{v}\|^2$.

9. **COMPONENTS OF ACCELERATION (REPRISE):**

$$\vec{a}(t) = \frac{d^2 s}{dt^2}\hat{T}(t) + \kappa(t)\left(\frac{ds}{dt}\right)^2 \hat{N}(t).$$

NOTE: There are two quadratic ideas happening here: a second derivative, d^2s/dt^2 in the direction of the motion and the square of the first derivative, $(ds/dt)^2$ in the normal direction. Coincidence?

10. **DEVELOPMENT OF TORSION:** Since acceleration (and hence, the force driving the motion) is contained completely in the osculating plane, and $\hat{B}$ is the direction of the osculating plane, we investigate how the direction of the osculating plane changes by looking at the rate of change of $\hat{B}$:

$$\begin{aligned} D_t[\hat{B}(t)] &= D_t[\hat{T}(t) \times \hat{N}(t)] \\ &= \hat{T}\,'(t) \times \hat{N}(t) + \hat{T}(t) \times \hat{N}\,'(t) \end{aligned}$$

Since $\hat{N}(t)$, by definition, is parallel to $\hat{T}\,'(t)$, $\hat{T}\,'(t) \times \hat{N}(t) = \vec{0}$, so $\hat{B}\,'(t) = \hat{T}(t) \times \hat{N}\,'(t)$. Therefore we know:

- $\hat{B}\,'(t)$ is orthogonal to $\hat{T}(t)$, since $\hat{B}\,'(t)$ is a cross product with $\hat{T}(t)$ as a factor
- $\hat{B}\,'(t)$ is orthogonal to $\hat{B}(t)$, since $\|\hat{B}(t)\| = 1$

Hence, $\hat{B}\,'(t)$ is parallel to $\hat{N}$. These relations hold for any choice of parameter, in particular for the 'natural' parameter, arc length. Hence we define the **torsion**, $\tau(s)$ as follows:

$$\frac{d\hat{B}}{ds} = -\tau(s)\hat{N}(s),$$

where the '$-$' is there as a convention only. By 'dotting' both sides with $\hat{N}(s)$, we get:

$$\frac{d\hat{B}}{ds} \cdot \hat{N}(s) = \left(-\tau(s)\hat{N}(s)\right) \cdot \hat{N}(s) = -\tau(s)\left(\hat{N}(s) \cdot \hat{N}(s)\right) = -\tau(s),$$

Which leads to our 'definition' of torsion as:

$$\tau = -\frac{d\hat{B}}{ds} \cdot \hat{N}(s) = -\frac{1}{\|\vec{v}(t)\|}\frac{d\hat{B}}{dt} \cdot \hat{N}(t),$$

the latter equation coming from the chain rule. Note that:

$$\|\tau\| = \left\|-\frac{d\hat{B}}{ds} \cdot \hat{N}(s)\right\| = \left\|\frac{d\hat{B}}{ds}\right\| \left\|\hat{N}\right\| \cos(0 \text{ or } \pi) = \left\|\frac{d\hat{B}}{ds}\right\|,$$

so torsion really does measure the tendency for the osculating plane to 'twist.'

11. **ALTERNATE FORMULAS FOR THE BINORMAL VECTOR AND TORSION:**

In our development of #7 above, we found:

$$\vec{a} \times \hat{T} = a_N \left(\hat{N} \times \hat{T}\right) = -a_N \hat{B},$$

so that

$$\hat{B} = -\frac{\vec{a} \times \hat{T}}{a_N} = \frac{\hat{T} \times \vec{a}}{a_N}$$

Putting $\hat{T} = \dfrac{\vec{v}}{\|\vec{v}\|}$ and $a_N = \dfrac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|}$, we get, after some cancellation:

$$\hat{B} = \frac{\vec{v} \times \vec{a}}{\|\vec{v} \times \vec{a}\|}.$$

We know from our development in #10 that $\hat{B}\,' = \hat{T} \times \hat{N}\,'$, so

$$\begin{aligned}
\tau &= -\frac{1}{\|\vec{v}\|}\frac{d\hat{B}}{dt} \cdot \hat{N} = -\frac{1}{\|\vec{v}\|}\left(\hat{T} \times \hat{N}\,'\right) \cdot \hat{N} \\
&= \frac{1}{\|\vec{v}\|}\left(\hat{N}\,' \times \hat{T}\right) \cdot \hat{N} = \frac{1}{\|\vec{v}\|}\hat{N}\,' \cdot \left(\hat{T} \times \hat{N}\right) \quad \text{(Triple Scalar Product Property)} \\
&= \frac{1}{\|\vec{v}\|}\left(\hat{N}\,' \cdot \hat{B}\right)
\end{aligned}$$

To get a handle on $\hat{N}\,' \cdot \hat{B}$, we take derivatives of both sides of $\vec{a} = a_T\hat{T} + a_N\hat{N}$:

$$\vec{a}\,' = a_T\,'\hat{T} + a_T\hat{T}\,' + a_N\,'\hat{N} + a_N\hat{N}\,'$$

and 'dot' both sides with $\hat{B}$:

$$\vec{a}\,' \cdot \hat{B} = a_T\,'\left(\hat{T} \cdot \hat{B}\right) + a_T\left(\hat{T}\,' \cdot \hat{B}\right) + a_N\,'\left(\hat{N} \cdot \hat{B}\right) + a_N\left(\hat{N}\,' \cdot \hat{B}\right)$$

Since $\hat{T}$ and $\hat{N}$ are orthogonal to $\hat{B}$, $\hat{T} \cdot \hat{B} = \hat{N} \cdot \hat{B} = 0$. Furthermore, since $\hat{T}\,'$ is parallel to $\hat{N}$, $\hat{T}\,' \cdot \hat{B} = 0$, too. Hence, we are left with:

$$\vec{a}\,' \cdot \hat{B} = a_N\left(\hat{N}\,' \cdot \hat{B}\right) \quad \text{or} \quad \hat{N}\,' \cdot \hat{B} = \frac{\vec{a}\,' \cdot \hat{B}}{a_N}$$

Substituting $a_N = \dfrac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|}$ and $\hat{B} = \dfrac{\vec{v} \times \vec{a}}{\|\vec{v} \times \vec{a}\|}$, we get:

$$\tau = \frac{1}{\|\vec{v}\|}\left(\hat{N}\,' \cdot \hat{B}\right) = \frac{\vec{a}\,' \cdot (\vec{v} \times \vec{a})}{\|\vec{v} \times \vec{a}\|^2} = \frac{(\vec{v} \times \vec{a}) \cdot \vec{a}\,'}{\|\vec{v} \times \vec{a}\|^2}.$$

CURVE ANALYSIS FORMULA SUMMARY SHEET

- **POSITION:** $\vec{r}$

- **VELOCITY:** $\vec{v} = \vec{r}\,'$

- **ACCELERATION:** $\vec{a} = \vec{v}\,'$

- **PRINCIPAL UNIT TANGENT VECTOR** $\hat{T} = \dfrac{\vec{v}}{\|\vec{v}\|}$

- **PRINCIPAL UNIT NORMAL VECTOR:** $\hat{N} = \dfrac{\hat{T}\,'}{\|\hat{T}\,'\|}$

- **COMPONENTS OF ACCELERATION:** $\vec{a} = a_T \hat{T} + a_N \hat{N}$ where $a_T = \dfrac{\vec{v}\cdot\vec{a}}{\|\vec{v}\|}$ and $a_N = \dfrac{\|\vec{v}\times\vec{a}\|}{\|\vec{v}\|}$

- **PRINCIPAL BINORMAL VECTOR:** $\hat{B} = \hat{T}\times\hat{N} = \dfrac{\vec{v}\times\vec{a}}{\|\vec{v}\times\vec{a}\|}$

- **CURVATURE:** $\kappa = \left\|\dfrac{d\hat{T}}{ds}\right\| = \dfrac{d\hat{T}}{ds}\cdot\hat{N} = \dfrac{\|\vec{v}\times\vec{a}\|}{\|\vec{v}\|^3}$

  NOTE: $\dfrac{d\hat{T}}{ds} = \kappa\hat{N}$

- **TORSION:** $\tau = -\dfrac{d\hat{B}}{ds}\cdot\hat{N} = \dfrac{(\vec{v}\times\vec{a})\cdot\vec{a}\,'}{\|\vec{v}\times\vec{a}\|^2}$.

  NOTE: $\dfrac{d\hat{B}}{ds} = -\tau\hat{N}$

- **FRENET - SERRET EQUATIONS:**

  From $\dfrac{d\hat{T}}{ds} = \kappa\hat{N}$ and $\dfrac{d\hat{B}}{ds} = -\tau\hat{N}$, we can differentiate $\hat{N} = \hat{B}\times\hat{T}$, to get:

$$\begin{cases} \dfrac{d\hat{T}}{ds} &= \kappa\,\hat{N} \\[2ex] \dfrac{d\hat{N}}{ds} &= -\kappa\,\hat{T} + \tau\,\hat{B} \\[2ex] \dfrac{d\hat{B}}{ds} &= -\tau\,\hat{N} \end{cases}$$